MATH2060 TUT05

9. If $f \in \mathcal{R}[a, b]$ and if $(\dot{\mathcal{P}}_n)$ is any sequence of tagged partitions of $[a, b]$ such that $||\dot{\mathcal{P}}_n|| \to 0$, prove that $\int_a^b f = \lim_n S(f; \dot{\mathcal{P}}_n)$.

Ans: Let $\varepsilon > 0$.

Since $f \in \mathcal{R}[a,b]$, $\exists\, \delta > 0$ s.t.

$\forall$ tagged partition $\dot{P}$ of $[a,b]$ with $||\dot{P}|| < \delta$,

$$\left| S(f; \dot{P}) - \int_a^b f \right| < \varepsilon.$$

Now, if $(\dot{P}_n)$ is any seq of tagged partition of $[a,b]$ s.t. $||\dot{P}_n|| \to 0$, then $\exists\, N \in \mathbb{N}$ s.t. $||\dot{P}_n|| < \delta$ $\forall\, n \geq N$.

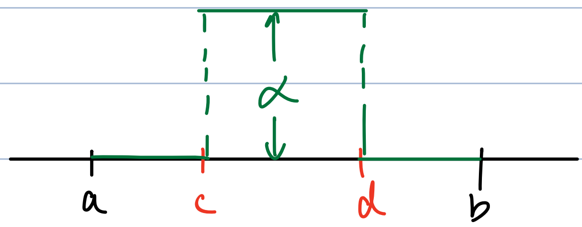
Thus, $\forall\, n \geq N$,

$$\left| S(f; \dot{P}_n) - \int_a^b f \right| < \varepsilon$$

Therefore $$\lim_{n\to\infty} S(f; \dot{P}_n) = \int_a^b f$$

13. Suppose that $c \le d$ are points in $[a, b]$. If $\varphi : [a, b] \to \mathbb{R}$ satisfies $\varphi(x) = \alpha > 0$ for $x \in [c, d]$ and $\varphi(x) = 0$ elsewhere in $[a, b]$, prove that $\varphi \in \mathcal{R}[a, b]$ and that $\int_a^b \varphi = \alpha(d - c)$. [*Hint*: Given $\varepsilon > 0$ let $\delta_\varepsilon := \varepsilon/4\alpha$ and show that if $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$ then we have $\alpha(d - c - 2\delta_\varepsilon) \le S(\varphi; \dot{\mathcal{P}}) \le \alpha(d - c + 2\delta_\varepsilon)$.]

Ans:



Given $\varepsilon > 0$, let $\delta_\varepsilon := \varepsilon/4\alpha$.
Suppose $\dot{P} := \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ is a tagged partition of $[a,b]$ with $\|\dot{P}\| < \delta_\varepsilon$.

Recall:

5. Let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ be a tagged partition of $[a, b]$ and let $c_1 < c_2$.
(a) If u belongs to a subinterval I_i whose tag satisfies $c_1 \le t_i \le c_2$, show that $c_1 - \|\dot{\mathcal{P}}\| \le u \le c_2 + \|\dot{\mathcal{P}}\|$.
(b) If $v \in [a, b]$ and satisfies $c_1 + \|\dot{\mathcal{P}}\| \le v \le c_2 - \|\dot{\mathcal{P}}\|$, then the tag t_i of any subinterval I_i that contains v satisfies $t_i \in [c_1, c_2]$.

a) : $u \in \bigcup_{c_1 \le t_i \le c_2} I_i \Rightarrow u \in [c_1 - \|\dot{P}\|, c_2 + \|\dot{P}\|]$.

b) : $v \in [c_1 + \|\dot{P}\|, c_2 - \|\dot{P}\|] \Rightarrow v \in \bigcup_{c_1 \le t_i \le c_2} I_i$

Now, $$S(\varphi; \dot{P}) = \sum_{t_i \in [c,d]} \overset{\alpha}{\cancel{\varphi(t_i)}}(x_i - x_{i-1}) + \sum_{t_i \notin [c,d]} \overset{0}{\cancel{\varphi(t_i)}}(x_i - x_{i-1})$$
$$\le \alpha\left[(d + \|\dot{P}\|) - (c - \|\dot{P}\|)\right]$$
$$\le \alpha(d - c + 2\delta_\varepsilon)$$

and $$S(\varphi; \dot{P}) \ge \alpha\left[(d - \|\dot{P}\|) - (c + \|\dot{P}\|)\right]$$
$$\ge \alpha(d - c - 2\delta_\varepsilon)$$

Hence $$|S(\varphi; \dot{P}) - \alpha(d - c)| \le 2\alpha\delta_\varepsilon = \varepsilon/2 < \varepsilon.$$

Therefore $\varphi \in \mathcal{R}[a,b]$ and $\int_a^b \varphi = \alpha(d - c)$

14. Let $0 \le a < b$, let $Q(x) := x^2$ for $x \in [a, b]$ and let $\mathcal{P} := \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[a, b]$. For each i, let q_i be the positive *square root* of

$$\tfrac{1}{3}\left(x_i^2 + x_i x_{i-1} + x_{i-1}^2\right).$$

(a) Show that q_i satisfies $0 \le x_{i-1} \le q_i \le x_i$.
(b) Show that $Q(q_i)(x_i - x_{i-1}) = \tfrac{1}{3}\left(x_i^3 - x_{i-1}^3\right)$.
(c) If $\dot{\mathcal{Q}}$ is the tagged partition with the same subintervals as $\mathcal{P}$ and the tags q_i, show that $S(Q; \dot{\mathcal{Q}}) = \tfrac{1}{3}\left(b^3 - a^3\right)$.
(d) Use the argument in Example 7.1.4(c) to show that $Q \in \mathcal{R}[a, b]$ and

$$\int_a^b Q = \int_a^b x^2\, dx = \tfrac{1}{3}\left(b^3 - a^3\right).$$

Ans: a) Since $0 \leqslant x_{i-1} < x_i$, we have

$$x_{i-1}^2 \leqslant q_i^2 = \frac{1}{3}\left(x_i^2 + x_i x_{i-1} + x_{i-1}^2\right) \leqslant \frac{1}{3}\left(x_i^2 + x_i^2 + x_i^2\right) = x_i^2$$

$$\Rightarrow \qquad x_{i-1} \leqslant q_i \leqslant x_i$$

b) $$Q(q_i)(x_i - x_{i-1}) = q_i^2 (x_i - x_{i-1}) = \frac{1}{3}\left(x_i^2 + x_i x_{i-1} + x_{i-1}^2\right)(x_i - x_{i-1})$$
$$= \frac{1}{3}\left(x_i^3 - x_{i-1}^3\right)$$

c) $$S(Q; \dot{Q}) = \sum_{i=1}^n Q(q_i)(x_i - x_{i-1})$$
$$= \frac{1}{3}\sum_{i=1}^n \left(x_i^3 - x_{i-1}^3\right)$$
$$= \frac{1}{3}\left(x_n^3 - x_0^3\right) = \frac{1}{3}\left(b^3 - a^3\right)$$

d) Let $\varepsilon > 0$. Take $\delta := \dfrac{\varepsilon}{4b(b-a)}$.
Suppose $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ is any tagged partition of $[a,b]$ with $\|\dot{P}\| < \delta$
Let $\dot{Q}$ be defined as in c).
Then $$|Q(t_i) - Q(q_i)| = |t_i^2 - q_i^2| = |t_i - q_i|\,(t_i + q_i)$$
$$\leqslant \delta \cdot (2b) \qquad (\because t_i, q_i \in [x_{i-1}, x_i])$$
and hence $$|S(Q; \dot{P}) - \cancel{S(Q; \dot{Q})}\ \tfrac{1}{3}(b^3 - a^3)| \leqslant \sum_{i=1}^n |Q(t_i) - Q(q_i)|\,(x_i - x_{i-1})$$
$$\leqslant 2\delta b(b-a) < \varepsilon$$
Therefore $Q \in \mathcal{R}[a,b]$ and $\int_a^b Q = \frac{1}{3}(b^3 - a^3)$ //

1. Let $f : [a,b] \to \mathbb{R}$. Show that $f \notin \mathcal{R}[a,b]$ if and only if there exists $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$ there exist tagged partitions $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ with $||\dot{\mathcal{P}}_n|| < 1/n$ and $||\dot{\mathcal{Q}}_n|| < 1/n$ such that $|S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{Q}}_n)| \geq \varepsilon_0$. (#)

Thm 7.2.1 (Cauchy Criterion)

A fcn $f : [a,b] \to \mathbb{R}$ belongs to $\mathcal{R}[a,b]$ iff $\forall \varepsilon > 0$, $\exists \eta_\varepsilon > 0$ s.t.

if $\dot{P}, \dot{Q}$ are tagged partitions of $[a,b]$ with $\|\dot{P}\|, \|\dot{Q}\| < \eta_\varepsilon$, then $|S(f;\dot{P}) - S(f;\dot{Q})| < \varepsilon$

"$\Leftarrow$" Suppose (#) holds.

Then $\exists \varepsilon_0 > 0$ s.t. $\forall n \in \mathbb{N}$, $\exists$ tagged partition $\dot{P}_n, \dot{Q}_n$ with $\|\dot{P}_n\| < 1/n$, $\|\dot{Q}_n\| < 1/n$ s.t. $|S(f;\dot{P}_n) - S(f;\dot{Q}_n)| \geq \varepsilon_0$

In particular, $\forall \eta > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $1/n_0 < \eta$ and so $\|\dot{P}_{n_0}\| < \eta$, $\|\dot{Q}_{n_0}\| < \eta$ and $|S(f;\dot{P}_n) - S(f;\dot{Q}_n)| \geq \varepsilon_0$

By Cauchy Criterion, $f \notin \mathcal{R}[a,b]$.

"$\Rightarrow$" Suppose $f \notin \mathcal{R}[a,b]$.

By Cauchy Criterion, $\exists \varepsilon_0 > 0$ s.t. $\forall \eta > 0$, $\exists$ tagged partition $\dot{P}, \dot{Q}$ s.t. $\|\dot{P}\| < \eta$, $\|\dot{Q}\| < \eta$ but $|S(f;\dot{P}) - S(f;\dot{Q})| \geq \varepsilon_0$

By taking $\eta = 1/n$, $n \in \mathbb{N}$, we obtain the sequences $\{\dot{P}_n\}$, $\{\dot{Q}_n\}$ with the desired properties

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Example Let a function $f\colon [0, \frac{\pi}{2}] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \cos^2 x, & x \in [0, \frac{\pi}{2}] \cap \mathbb{Q} \\ 0, & \text{else.} \end{cases}$$

Is this function Riemann integrable?

Ans: Try to apply Cauchy Criterion to show that $f \notin \mathcal{R}[0, \frac{\pi}{2}]$.

Let $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ be a partition of $[0, \frac{\pi}{2}]$

Let $\dot{\mathcal{P}}_1 = \{[x_{i-1}, x_i], q_i\}_{i=1}^n$ where $q_i \in [x_{i-1}, x_i] \cap \mathbb{Q}$
$\dot{\mathcal{P}}_2 = \{[x_{i-1}, x_i], r_i\}_{i=1}^n$ where $r_i \in [x_{i-1}, x_i] \setminus \mathbb{Q}$ } exist by density of $\mathbb{Q}$, $\mathbb{R}\setminus\mathbb{Q}$.

Suppose $x_{k-1} < \frac{\pi}{4} \leq x_k$. Then

$$\begin{aligned} S(f; \dot{\mathcal{P}}_1) &= \sum_{i=1}^n f(q_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n \cos^2(q_i)(x_i - x_{i-1}) \\ &\geq \sum_{i=1}^{k-1} \frac{1}{2}(x_i - x_{i-1}) \\ &= \frac{1}{2}(x_{k-1} - x_0) \\ &= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}\left(\frac{\pi}{4} - x_{k-1}\right) \\ &\geq \frac{\pi}{8} - \frac{1}{2}\|\dot{\mathcal{P}}_1\| \end{aligned}$$

$\left(\because q_1 \leq \cdots \leq q_{k-1} \leq x_{k-1} < \frac{\pi}{4} \quad \therefore \cos^2(q_i) \geq \cos^2 \frac{\pi}{4} = \frac{1}{2} \right)$

and

$$S(f; \dot{\mathcal{P}}_2) = \sum_{i=1}^n \underbrace{\cancel{f(r_i)}}_{0}(x_i - x_{i-1}) = 0$$

Take $\varepsilon_0 = \frac{\pi}{16}$. Let $\eta > 0$.
Find a uniform partition $\mathcal{P}$ with $\|\mathcal{P}\| < \min\{\eta, \frac{\pi}{8}\}$
Then $\dot{\mathcal{P}}_1, \dot{\mathcal{P}}_2$ defined above satisfy

$$\|\dot{\mathcal{P}}_1\| = \|\dot{\mathcal{P}}_2\| = \|\mathcal{P}\| < \eta$$

but $\left| S(f; \dot{\mathcal{P}}_1) - S(f; \dot{\mathcal{P}}_2) \right| \geq \frac{\pi}{8} - \frac{1}{2}\left(\frac{\pi}{8}\right) = \frac{\pi}{16} = \varepsilon_0$.

By Cauchy Criterion 7.2.1, $f \notin \mathcal{R}[0, \frac{\pi}{2}]$ //